PAPER

Computation of Floquet Multipliers Using an Iterative Method for Variational Equations

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SUMMARY This paper proposes a new method to numerically obtain Floquet multipliers which characterize stability of periodic orbits of ordinary differential equations. For sufficiently smooth periodic orbits, we can compute Floquet multipliers using some standard numerical methods with enough accuracy. However, it has been reported that these methods may produce incorrect results under some conditions. In this work, we propose a new iterative method to compute Floquet multipliers using eigenvectors of matrix solutions of the variational equations. Numerical examples show effectiveness of the proposed method.

key words: *numerical method, stability, ordinary differential equations, periodic orbits, Floquet multipliers*

1. Introduction

This paper studies numerical methods for stability analysis of periodic orbits of ordinary differential equations given by

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(t, \boldsymbol{x}) \quad \text{with} \quad \boldsymbol{f}(t+T, \boldsymbol{x}) = \boldsymbol{f}(t, \boldsymbol{x}), \tag{1}$$

where T denotes the period, $\boldsymbol{x} \in \mathbb{R}^N$, and $\boldsymbol{f} : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is of class C^2. This is a non-autonomous system and its solution with an initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ can be written in the form

$$\boldsymbol{x}(t) = \varphi_{t_0}^{t-t_0}(\boldsymbol{x}_0)\,. \tag{2}$$

A periodic solution with the period T satisfies

$$\boldsymbol{x}(t_0 + T) = \varphi_{t_0}^{T}(\boldsymbol{x}_0) = \boldsymbol{x}_0\,. \tag{3}$$

Time variation of a small perturbation $\boldsymbol{v}(t)$ of this periodic solution $\boldsymbol{x}(t)$ is governed by

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{t,\ \boldsymbol{x}=\varphi_{t_0}^{t-t_0}(\boldsymbol{x}_0)} \boldsymbol{v}(t) \quad \text{with} \quad \boldsymbol{v}(t_0) = \boldsymbol{v}_0\,. \tag{4}$$

The solution $\boldsymbol{v}(t)$ of (4) can be expressed as

$$\boldsymbol{v}(t) = \boldsymbol{X}_0(\tau)\,\boldsymbol{v}_0 \tag{5}$$

with the matrix solution $\boldsymbol{X}_0(\tau) \in \mathbb{R}^{N\times N}$ $(\tau = t - t_0)$ of the variational equations given by

$$\frac{\mathrm{d}\boldsymbol{X}_0}{\mathrm{d}\tau} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{t=t_0+\tau,\ \boldsymbol{x}=\varphi_{t_0}^{\tau}(\boldsymbol{x}_0)} \boldsymbol{X}_0(\tau) \ \text{ with } \ \boldsymbol{X}_0(0) = I\,, \tag{6}$$

where I denotes the identity matrix. Eigenvalues λ_j $(j = 1, 2, \cdots, N)$ of the matrix solution $\boldsymbol{X}_0(T)$ of (6) determine the stability of the periodic orbit. These eigenvalues are called Floquet multipliers.

There are various numerical methods to solve periodic orbits of (1) such as the shooting method and the collocation method [3], [8], and some standard bifurcation analysis tools such as AUTO [4] are easily available. On the other hand, it has been reported that these commonly used numerical methods can incorrectly catch stability of periodic orbits under some conditions, even if the periodic orbits were accurately computed [5], [9], [10].

For example, Fig. 1 shows computed results of Floquet multipliers λ_1 and λ_2 of the nonlinear Mathieu equations [7]

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y\,, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -(1 + p\cos t)\,\sin x\,, \end{aligned} \tag{7}$$

using one of conventional methods "Method 1" which is summarized in Sect. 2. We can find erroneous results in the smaller Floquet multiplier λ_2 for the parameter $p > 20$ in Fig. 1. It should be noted that the ratio $|\lambda_1/\lambda_2|$ drastically increases with p for $p < 20$. Lust indicated this type of trouble to compute Floquet multipliers and developed an improved algorithm [9]. His method divides a periodic orbit into some sub-orbits, and computes the expanding or contracting rates of perturbations on each sub-orbit using the periodic Schur decomposition [2] of $\boldsymbol{X}_0(T)$. Finally, we can get the Floquet multipliers by multiplying those rates.

In Lust's method [9] and some conventional methods, the variational Eqs. (6) are solved as initial value problems in which it is hard to control numerical errors. In the previous work [12], we showed that such a numerical method for (6) may cause some troubles to obtain accurate Floquet multipliers. Also, when the ratio of eigenvalues of the matrix $\boldsymbol{X}_0(T)$ is large as shown in Fig. 1, some elements of $\boldsymbol{X}_0(T)$ can be considerably large, and round-off errors may affect numerical computation of Floquet multipliers.

The basic ideas of this work are to construct an iterative method for the variational equations (6), and to use eigenvectors of $\boldsymbol{X}_0(T)$ as the initial condition of (4) or (6). The iterative method enables us to control accuracy of computed results of the variational equations. The initial condition $\boldsymbol{X}_0(0)$ of (6) can be set to any non-singular matrix.

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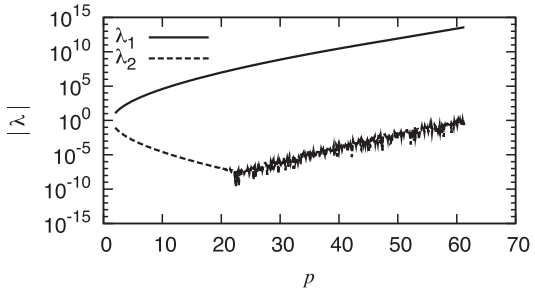


Fig. 1 The Floquet multipliers λ_j ($j = 1, 2$) computed with the conventional method "Method 1" for the Mathieu equations (7). p: the parameter of the equations.

Suitable choice of the initial condition $\boldsymbol{X}_0(0)$ reduces errors of numerical computation of eigenvalues of $\boldsymbol{X}_0(T)$, namely, Floquet multipliers.

Section 2 summarizes a standard method of computation of periodic orbits, namely, the multiple shooting method. Section 3 proposes a new iterative method to compute Floquet multipliers using eigenvectors of $\boldsymbol{X}_0(T)$. The proposed method can also be applied to autonomous systems with minor modifications. Section 3.3 describes the computational method for autonomous systems. Section 4 shows some results of numerical examples.

2. The Multiple Shooting Method for Periodic Orbits: Method 1 [3], [8], [14]

The multiple shooting method is one of standard methods to compute a periodic orbit of (1). This method divides the periodic orbit with the period T into $M+1$ sub-orbits such that

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}(t_{k+1}) = \varphi_{t_k}^{h_k}(\boldsymbol{x}_k) \text{ for } k = 0, 1, \cdots, M, \tag{8}$$

where $t_0 < t_1 < \cdots < t_M < t_{M+1} = t_0 + T$, $h_k = t_{k+1} - t_k$ and $\boldsymbol{x}_{M+1} = \boldsymbol{x}_0$. For non-autonomous systems (1), the period T is given and each time t_k can be fixed. Using an iterative method such as Newton's method for (8), we can obtain approximate solutions of $\boldsymbol{x}_k$.

Writing $\tau = t - t_k$, we can express the variational equations (6) for each sub-orbit as

$$\frac{\mathrm{d}\boldsymbol{X}_k}{\mathrm{d}\tau} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{t=t_k+\tau,\, \boldsymbol{x}=\varphi_{t_k}^{\tau}(\boldsymbol{x}_k)} \boldsymbol{X}_k(\tau) \quad \text{with } \boldsymbol{X}_k(0) = I\,, \tag{9}$$

for $k = 0, 1, \cdots, M$ where $\boldsymbol{X}_k(\tau) \in \mathbb{R}^{N\times N}$. Floquet multipliers are eigenvalues of the matrix solution $\boldsymbol{X}_0(T)$ given by

$$\boldsymbol{X}_0(T) = \boldsymbol{X}_M(h_M) \cdots \boldsymbol{X}_1(h_1)\, \boldsymbol{X}_0(h_0)\,. \tag{10}$$

We can numerically get $\varphi_{t_k}^{h_k}(\boldsymbol{x}_k)$ and $\boldsymbol{X}_k(h_k)$ using the Runge-Kutta method, and the Floquet multipliers using standard tools for linear algebra such as LAPACK [1].

This is one of the commonly used methods for stability analysis of periodic orbits using the multiple shooting method. Hereafter we call this method "Method 1."

3. The Proposed Method: Method 2

This section proposes a computational method to iteratively solve variational equations and obtain Floquet multipliers. The ideas are to construct an iterative method for the variational equations using a property of solutions of differential equations, and to use eigenvectors of $\boldsymbol{X}_0(T)$ as the initial condition of (4) or (6).

3.1 An Iterative Method for the Variational Equations [12]

The map $\varphi_{t_k}^{\tau}$ in (2) has the following property

$$\varphi_{t_k}^{\tau+\sigma} = \varphi_{t_k+\tau}^{\sigma} \circ \varphi_{t_k}^{\tau} \quad \text{for} \quad \tau,\, \sigma \in \mathbb{R}\,. \tag{11}$$

This property gives

$$\varphi_{t_k}^{h_k - s_k}(\boldsymbol{x}_k) = \varphi_{t_k+h_k}^{-s_k} \circ \varphi_{t_k}^{h_k}(\boldsymbol{x}_k) = \varphi_{t_{k+1}}^{-s_k}(\boldsymbol{x}_{k+1})\,, \tag{12}$$

where $s_k = \rho_k h_k$ with constant $\rho_k \in [0, 1]$ for $k = 0, 1, \cdots, M$. Then, it follows that

$$\boldsymbol{g}_k(\boldsymbol{x}_k, \boldsymbol{x}_{k+1}) := \varphi_{t_k}^{h_k - s_k}(\boldsymbol{x}_k) - \varphi_{t_{k+1}}^{-s_k}(\boldsymbol{x}_{k+1}) = \boldsymbol{0}\,. \tag{13}$$

Also, solutions of the variational equations (9) have the above property. Writing the solution as $\boldsymbol{X}_k(\tau) = \Psi_{t_k,\boldsymbol{x}_k}^{\tau}(\boldsymbol{X}_k(0))$ where the map $\Psi_{t_k,\boldsymbol{x}_k}^{\tau}$ satisfies $\Psi_{t_k,\boldsymbol{x}_k}^{\tau+\sigma} = \Psi_{t_k+\tau,\, \boldsymbol{x}(t_k+\tau)}^{\sigma} \circ \Psi_{t_k,\boldsymbol{x}_k}^{\tau}$ ($\tau, \sigma \in \mathbb{R}$), we can express the property as

$$\begin{aligned}&\boldsymbol{Q}_k(\boldsymbol{x}_k, \boldsymbol{x}_{k+1}, \boldsymbol{X}_k(h_k))\\ &\quad := \Psi_{t_k,\boldsymbol{x}_k}^{h_k - s_k}(\boldsymbol{X}_k(0)) - \Psi_{t_{k+1},\boldsymbol{x}_{k+1}}^{-s_k}(\boldsymbol{X}_k(h_k)) = O\,,\end{aligned} \tag{14}$$

where O denotes the null matrix and $\boldsymbol{X}_k(0)$ is set to the identity matrix $\boldsymbol{X}_k(0) = I$.

Introduction of s_k enables us to solve not only ordinary differential equations (1) but also variational equations (6) using an iterative method for $\boldsymbol{g}_k$ (13) and $\boldsymbol{Q}_k$ (14). We can iteratively get the approximate solutions $\tilde{\boldsymbol{x}}_k, \tilde{\boldsymbol{X}}_k(h_k)$ such that (13) and (14) are simultaneously satisfied with enough accuracy. Then the convergence conditions of the method are given by

$$d_1 := \max_{0\le k\le M} \|\tilde{\boldsymbol{g}}_k\| < \delta_1\,, \tag{15}$$

and

$$d_2 := \max_{0\le k\le M} \|\tilde{\boldsymbol{Q}}_k\| < \delta_2\,, \tag{16}$$

with sufficiently small $\delta_1, \delta_2 > 0$ and

$$\|\tilde{\boldsymbol{Q}}_k\| = \|(\tilde{\boldsymbol{q}}_1 \tilde{\boldsymbol{q}}_2 \cdots \tilde{\boldsymbol{q}}_N)\| := \max_{1\le j\le N} \|\tilde{\boldsymbol{q}}_j\|\,, \tag{17}$$

where $\tilde{\boldsymbol{g}}_k$ and $\tilde{\boldsymbol{Q}}_k$ denote the approximate solutions of (13) and (14), respectively, and $\tilde{\boldsymbol{q}}_j$ denotes the j-th column vector of the matrix $\tilde{\boldsymbol{Q}}_k$. We can compute $\tilde{\boldsymbol{g}}_k$ and $\tilde{\boldsymbol{Q}}_k$ using the Runge-Kutta method. These conditions can be used as indices for accuracy of solutions.

3.2 The Initial Condition of the Variational Equations

As stated in Sect. 1, the initial condition $\boldsymbol{X}_0(0)$ of the variational equations (6) can be set to any non-singular matrix. This section shows that errors of computing eigenvalues can be reduced using eigenvectors of $\boldsymbol{X}_0(T)$ as the initial condition.

Let $\boldsymbol{v}_{j,0} \in \mathbb{C}^N$ $(j = 1, 2, \cdots, N)$ denote the linearly independent normalized eigenvectors for the Floquet multipliers $\lambda_j \in \mathbb{C}$, namely,

$$\lambda_j \boldsymbol{v}_{j,0} = \boldsymbol{X}_0(T)\boldsymbol{v}_{j,0}\,. \tag{18}$$

Consider a solution $\boldsymbol{v}_j(t) = \psi^{t-t_0}_{t_0,\boldsymbol{x}_0}(\boldsymbol{v}_{j,0})$ of (4) with the initial value $\boldsymbol{v}_{j,0}$. Write solutions $\boldsymbol{v}_j(t)$ at $t = t_k$ as $\boldsymbol{v}_{j,k} := \boldsymbol{v}_j(t_k) = \psi^{t_k-t_0}_{t_0,\boldsymbol{x}_0}(\boldsymbol{v}_{j,0})$ for $k = 1, 2, \cdots, M+1$. Then $\boldsymbol{v}_{j,k}$ and $\boldsymbol{v}_{j,k-1}$ are related by

$$\boldsymbol{v}_{j,k} = \psi^{h_{k-1}}_{t_{k-1},\boldsymbol{x}_{k-1}}(\boldsymbol{v}_{j,k-1}) = \boldsymbol{X}_{k-1}(h_{k-1})\boldsymbol{v}_{j,k-1}\,, \tag{19}$$

where $\boldsymbol{X}_k(h_k)$ is a solution of (9). From this we get

$$\boldsymbol{v}_{j,k} = \boldsymbol{X}_{k-1}(h_{k-1})\boldsymbol{X}_{k-2}(h_{k-2})\cdots\boldsymbol{X}_0(h_0)\boldsymbol{v}_{j,0}\,. \tag{20}$$

Furthermore, using (10) and (18), we can obtain

$$\begin{aligned}\lambda_j \boldsymbol{v}_{j,k} &= \lambda_j\, \boldsymbol{X}_{k-1}(h_{k-1})\cdots\boldsymbol{X}_0(h_0)\,\boldsymbol{v}_{j,0}\\ &= \boldsymbol{X}_{k-1}(h_{k-1})\cdots\boldsymbol{X}_0(h_0)\boldsymbol{X}_0(T)\,\boldsymbol{v}_{j,0}\\ &= \boldsymbol{X}_{k-1}(h_{k-1})\cdots\boldsymbol{X}_0(h_0)\boldsymbol{X}_M(h_M)\cdots\boldsymbol{X}_k(h_k)\\ &\quad\ \boldsymbol{X}_{k-1}(h_{k-1})\cdots\boldsymbol{X}_0(h_0)\,\boldsymbol{v}_{j,0}\\ &= \boldsymbol{X}_k(T)\,\boldsymbol{v}_{j,k}\,.\end{aligned} \tag{21}$$

Here the following relation is used

$$\boldsymbol{X}_k(T) = \boldsymbol{X}_{k-1}(h_{k-1})\cdots\boldsymbol{X}_0(h_0)\boldsymbol{X}_M(h_M)\cdots\boldsymbol{X}_k(h_k)\,. \tag{22}$$

From these, we can see that $\boldsymbol{v}_{j,k}$ is an eigenvector for an eigenvalue λ_j of $\boldsymbol{X}_k(T)$.

Since $\boldsymbol{v}_j(t_0 + T) = \boldsymbol{X}_0(T)\boldsymbol{v}_{j,0}$, it follows from (18) that

$$\lambda_j \boldsymbol{v}_{j,0} = \boldsymbol{v}_j(t_0 + T) = \boldsymbol{v}_{j,\,M+1} = \psi^{h_M}_{t_M,\boldsymbol{x}_M}(\boldsymbol{v}_{j,M})\,, \tag{23}$$

and

$$\lambda_j = \langle\, \boldsymbol{v}_{j,0},\ \boldsymbol{v}_j(t_0 + T)\,\rangle = \langle\, \boldsymbol{v}_{j,0},\ \psi^{h_M}_{t_M,\boldsymbol{x}_M}(\boldsymbol{v}_{j,M})\,\rangle\,, \tag{24}$$

where $\langle\cdot,\ \cdot\rangle$ denotes the inner product. Using (24), we can compute the Floquet multiplier λ_j with relatively small round-off errors, even if some elements of the matrix $\boldsymbol{X}_0(T)$ are considerably large. It is because (24) does not require transformation of $\boldsymbol{X}_0(T)$ which is commonly used in computation of eigenvalues of a matrix and may cause round-off errors.

In addition, in order to avoid overflow of computed results, we normalize $\boldsymbol{v}_j(t)$ at every $t = t_k$ as $\boldsymbol{u}_{j,k} = \boldsymbol{v}_{j,k}/\|\boldsymbol{v}_{j,k}\|$. This is similar to the computation of Lyapunov exponents [15]. Then $\boldsymbol{u}_{j,k+1}$ can be expressed as

$$\begin{aligned}\boldsymbol{u}_{j,k+1} &= \frac{\boldsymbol{v}_{j,k+1}}{\|\boldsymbol{v}_{j,k+1}\|} = \frac{\boldsymbol{v}_{j,k+1}}{\|\psi^{h_k}_{t_k,\boldsymbol{x}_k}(\boldsymbol{v}_{j,k})\|}\\ &= \frac{\boldsymbol{v}_{j,k+1}}{\|\boldsymbol{v}_{j,k}\|\|\psi^{h_k}_{t_k,\boldsymbol{x}_k}(\boldsymbol{u}_{j,k})\|}\,.\end{aligned} \tag{25}$$

Thus (24) can be rewritten in the form

$$\begin{aligned}\lambda_j &= \left\langle \boldsymbol{u}_{j,0},\ \psi^{h_M}_{t_M,\boldsymbol{x}_M}(\boldsymbol{u}_{j,M}) \right\rangle \|\,\boldsymbol{v}_{j,M}\,\|\\ &= \left\langle \boldsymbol{u}_{j,0},\ \frac{\psi^{h_M}_{t_M,\boldsymbol{x}_M}(\boldsymbol{u}_{j,M})}{\|\psi^{h_M}_{t_M,\boldsymbol{x}_M}(\boldsymbol{u}_{j,M})\|} \right\rangle \|\psi^{h_M}_{t_M,\boldsymbol{x}_M}(\boldsymbol{u}_{j,M})\|\\ &\qquad \|\,\psi^{h_{M-1}}_{t_{M-1},\boldsymbol{x}_{M-1}}(\boldsymbol{u}_{j,M-1})\,\|\,\|\,\boldsymbol{v}_{j,M-1}\,\|\\ &= \left\langle \boldsymbol{u}_{j,0},\ \frac{\psi^{h_M}_{t_M,\boldsymbol{x}_M}(\boldsymbol{u}_{j,M})}{\|\psi^{h_M}_{t_M,\boldsymbol{x}_M}(\boldsymbol{u}_{j,M})\|} \right\rangle \prod_{k=0}^{M} \|\psi^{h_k}_{t_k,\boldsymbol{x}_k}(\boldsymbol{u}_{j,k})\|\,.\end{aligned} \tag{26}$$

In order to obtain accurate Floquet multipliers using (26), we modify the method in Sect. 3.1 so that the normalized eigenvectors $\boldsymbol{u}_{j,k}$ of $\boldsymbol{X}_k(T)$ are iteratively computed. Using the eigenvectors $\boldsymbol{u}_{j,k}$, we set the initial condition of (9) as

$$\hat{\boldsymbol{X}}_k(0) = \left(\boldsymbol{u}_{1,k}\ \boldsymbol{u}_{2,k}\ \cdots\ \boldsymbol{u}_{N,k}\right). \tag{27}$$

Then, equations (19), (23)–(25) yield

$$\begin{aligned}\hat{\boldsymbol{X}}_k(h_k) &= \Psi^{h_k}_{t_k,\boldsymbol{x}_k}(\hat{\boldsymbol{X}}_k(0))\\ &= \left(\psi^{h_k}_{t_k,\boldsymbol{x}_k}(\boldsymbol{u}_{1,k})\,\psi^{h_k}_{t_k,\boldsymbol{x}_k}(\boldsymbol{u}_{2,k})\ \cdots\ \psi^{h_k}_{t_k,\boldsymbol{x}_k}(\boldsymbol{u}_{N,k})\right)\\ &= \left(\boldsymbol{u}_{1,k+1}\ \boldsymbol{u}_{2,k+1}\ \cdots\ \boldsymbol{u}_{N,k+1}\right) D_k\\ &= \hat{\boldsymbol{X}}_{k+1}(0)\,D_k \quad \text{for} \quad k = 0, 1, \cdots, M\,,\end{aligned} \tag{28}$$

where $\hat{\boldsymbol{X}}_{M+1}(0) = \hat{\boldsymbol{X}}_0(0)$ and $D_k \in \mathbb{R}^{N\times N}$ are the diagonal matrices such that

$$D_k = \operatorname*{diag}_{1\le j\le N} \|\psi^{h_k}_{t_k,\boldsymbol{x}_k}(\boldsymbol{u}_{j,k})\|\,, \tag{29}$$

for $k = 0, 1, \cdots, M-1$ and

$$D_M = \operatorname*{diag}_{1\le j\le N} \langle\, \boldsymbol{u}_{j,0},\ \psi^{h_M}_{t_M,\boldsymbol{x}_M}(\boldsymbol{u}_{j,M})\rangle\,. \tag{30}$$

Since $\boldsymbol{u}_{j,k}$ and $\psi^{h_k}_{t_k,\boldsymbol{x}_k}(\boldsymbol{u}_{j,k})$ are the j-th columns of $\hat{\boldsymbol{X}}_k(0)$ and $\Psi^{h_k}_{t_k,\boldsymbol{x}_k}(\hat{\boldsymbol{X}}_k(0))$, respectively, D_k depends on $\boldsymbol{x}_k$ and $\hat{\boldsymbol{X}}_k(0)$. Equation (14) can be rewritten as

$$\begin{aligned}&\boldsymbol{Q}_k(\boldsymbol{x}_k, \boldsymbol{x}_{k+1}, \hat{\boldsymbol{X}}_k(0), \hat{\boldsymbol{X}}_{k+1}(0))\\ &\quad = \Psi^{h_k-s_k}_{t_k,\boldsymbol{x}_k}(\hat{\boldsymbol{X}}_k(0)) - \Psi^{-s_k}_{t_{k+1},\boldsymbol{x}_{k+1}}(\hat{\boldsymbol{X}}_{k+1}(0)D_k) = O\,.\end{aligned} \tag{31}$$

An iterative method for (13) and (31) can control errors of approximate solutions $\boldsymbol{x}_k$ and $\hat{\boldsymbol{X}}_k(0)$ with thresholds δ_1 (15) and δ_2 (16).

The proposed method can be summarized as follows: First, we compute the periodic solutions $\boldsymbol{x}_k$ using the iterative method for (13). Next, we can get Floquet multipliers λ_j $(j = 1, 2, \cdots, N)$ and $\hat{\boldsymbol{X}}_k(0)$, namely, the corresponding eigenvectors $\boldsymbol{u}_{j,k}$ using the above method. We call this method "Method 2."

3.3 Application to Autonomous Systems

The proposed method "Method 2" can be applied to autonomous systems given by

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}), \tag{32}$$

where $\boldsymbol{f} : \mathbb{R}^N \to \mathbb{R}^N$ is of class C^2. For autonomous systems, the time intervals $h_k = t_{k+1} - t_k$ $(k = 0, 1, \cdots, M)$ are unknown and the period $T = \sum_{k=0}^{M} h_k$. Since the initial time t_0 can be set arbitrarily, we set $t_0 = 0$. In order to determine the unknown variables h_k, $\boldsymbol{x}_k$ and $\hat{\boldsymbol{X}}_k(0)$, we add another condition under which each $\boldsymbol{x}_k$ is located on Poincaré sections H_k transverse to the periodic orbit. We write this condition as

$$b_k(\boldsymbol{x}_k) = 0 \quad \text{for} \quad k = 0, 1, \cdots, M. \tag{33}$$

We can iteratively obtain approximate solutions of $\{h_k,\ \boldsymbol{x}_k,\ \hat{\boldsymbol{X}}_k(0)\}_{k=0,1,\cdots,M}$ using Newton's method for (13) and (31) with (33). Using h_k, we can determine t_k and the period T.

4. Numerical Examples

This section shows some computed results using the proposed method "Method 2" for the Mathieu equations, the FitzHugh-Nagumo equations, and a mathematical model for plasma dynamics. In the following computations, s_k in (13) and (31) is fixed to $\frac{h_k}{2}$. We may expect that suitable choice of s_k can improve computed results, but this is left for future works. Eigenvalues and eigenvectors of $\boldsymbol{X}_k(T)$ were computed using "geev" of LAPACK [1] and were used as initial approximate solutions for "Method 2."

4.1 The Mathieu Equations

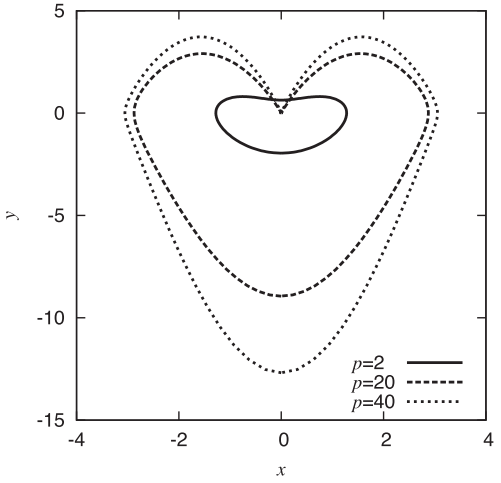
Figures 2(a) and (b) show the computed periodic orbits in the state space and the corresponding time series of the Mathieu equations (7) with $p = 2$, 20, and 40, respectively. The equilibrium point $\boldsymbol{x}_e$ is located at the origin in the state space. In these computations, $T = 2\pi$, $M{+}1 = 50$, $t_k = \frac{T}{M+1}k$, and $h_k = t_{k+1} - t_k = \frac{T}{M+1}$ $(k = 0, 1, \cdots, M)$. The convergence conditions (15) and (16) of "Method 2" were set to $\delta_1 = \delta_2 = 10^{-6}$ and the time increment Δt of the 4th order Runge-Kutta method was set to $\Delta t = 0.1$.

Figure 2 shows that the periodic orbit with large p moves near the equilibrium point $\boldsymbol{x}_e$ slowly, and away from $\boldsymbol{x}_e$ quickly. In this situation, the ratio of the Floquet multipliers $|\lambda_1/\lambda_2|$ $(|\lambda_1| \geq |\lambda_2|)$ becomes considerably large, and "Method 1" can produce incorrect results as shown in Fig. 1. On the other hand, the proposed method "Method 2" gives more accurate results as shown in Fig. 3.

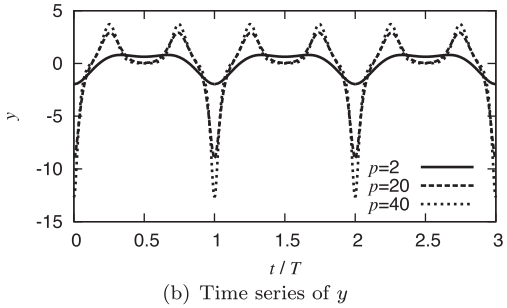
Equation (21) means the j-th Floquet multiplier λ_j (18) is equal to eigenvalues $\lambda_{j,k}$ of $\boldsymbol{X}_k(T)$ (22) for $k = 0, 1, \cdots, M$. However, numerical results of $\lambda_{j,k}$ can be different from each other. Thus we can use

$$R(\lambda_j) = \frac{1}{|\lambda_{j,0}|}\left|\max_{0\leq k\leq M}\lambda_{j,k} - \min_{0\leq k\leq M}\lambda_{j,k}\right|, \tag{34}$$

as an index of accuracy of the computed Floquet multipliers. Figures 4(a) and (b) show $R(\lambda_j)$ $(|\lambda_1| \geq |\lambda_2|)$ using "Method 1" and "Method 2", respectively. We can see that $R(\lambda_2)$ of "Method 1" drastically increases with p, but that both $R(\lambda_1)$ and $R(\lambda_2)$ of "Method 2" are almost zero. These results demonstrate effectiveness of the proposed method "Method 2".



(a) Periodic orbits in the state space



(b) Time series of y

Fig. 2 Computed results of the periodic orbits of the Mathieu equations (7) by "Method 2." T: the period. p: the parameter of the equations.

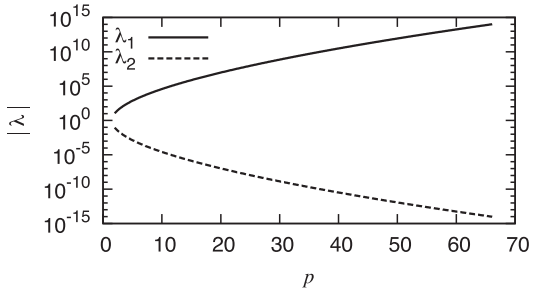
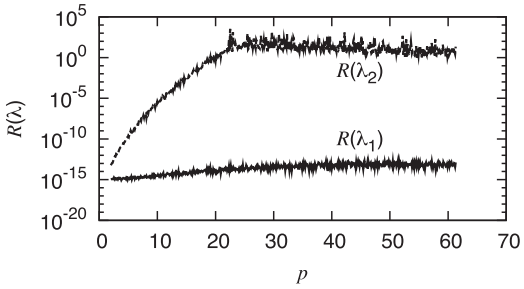


Fig. 3 The Floquet multipliers λ_j $(j = 1, 2)$ computed with "Method 2" for the Mathieu equations (7). p: the parameter of the equations.

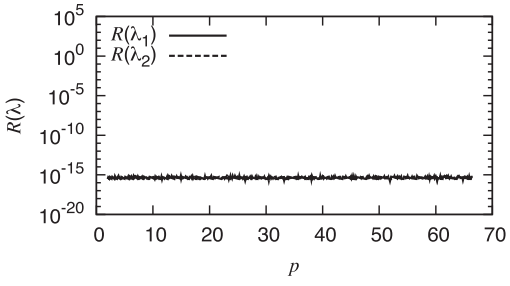
4.2 The FitzHugh-Nagumo Equations

Figures 5(a) and (b) show the computed periodic orbits in the state space and the corresponding time series of the FitzHugh-Nagumo equations [6], [11] given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y,$$



(a) Method 1



(b) Method 2

Fig. 4 Comparison of accuracy of the numerical results of the Floquet multipliers λ_j ($j = 1, 2$) for the Mathieu equations (7). (a) and (b) compare $R(\lambda_j)$ defined by (34) of "Method 1" and "Method 2." p: the parameter of the equations.

$$\frac{dy}{dt} = qy + x(x-1)(x-p) + z, \qquad (35)$$
$$\frac{dz}{dt} = \frac{r}{q}x,$$

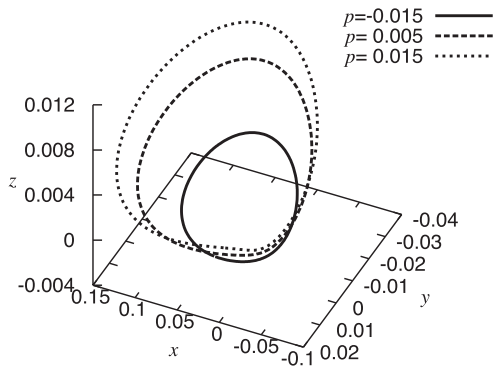
for $p = -0.015$, 0.01, and 0.015, respectively. This is a simple mathematical model of nerve axon dynamics. The equilibrium $\boldsymbol{x}_e$ is located at the origin in the state space. In these computations, q and r were fixed to $q = 0.28288600873$ and $r = 0.0025$, $M+1 = 25$, the time increment Δt of the 4th order Runge-Kutta method was set to $\Delta t = 0.1$, and the convergence conditions for "Method 2" were set to $\delta_1 = \delta_2 = 10^{-6}$.

Similarly to the computed results of the Mathieu equations, Figs. 5(a) and (b) show that the periodic orbits with large p slowly moves near the equilibrium $\boldsymbol{x}_e$ and quickly away from $\boldsymbol{x}_e$. Figure 6 shows the computed Floquet multipliers λ_1, λ_2 and λ_3 ($|\lambda_1| \geq |\lambda_2| \geq |\lambda_3|$) using "Method 2." We can see that the ratios $|\lambda_1/\lambda_2|$ and $|\lambda_1/\lambda_3|$ rapidly increase with p. It should be noted that one of the Floquet multipliers of this autonomous system must be equal to unity [10], and that, in this case, the corresponding eigenvalue λ_u is λ_2.

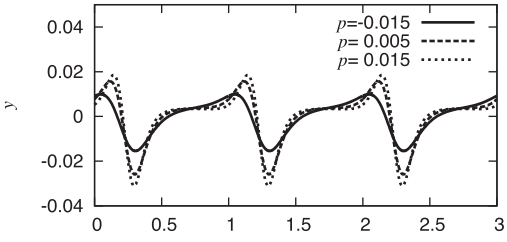
Figure 7 compares $R(\lambda_j)$ ($j = 1, 2, 3$) defined by (34) of "Method 1" and "Method 2." As shown in Sect. 4.1, $R(\lambda_j)$ can be used as an index of accuracy of numerical results. Figures 7(a) and (b) show that the proposed method "Method 2" gives more accurate results than "Method 1" in this example. Using "Method 2," $R(\lambda_j)$ can be further reduced with severe convergence conditions.

Theoretically, one of the Floquet multipliers λ_u must be equal to unity. Thus we can use

$$E(\lambda_u) := |\lambda_u - 1|, \qquad (36)$$



(a) Periodic orbits in the state space



(b) Time series of y

Fig. 5 Computed results of the periodic orbits of the FitzHugh-Nagumo equations (35) by "Method 2." T: the period. p: the parameter of the equations.

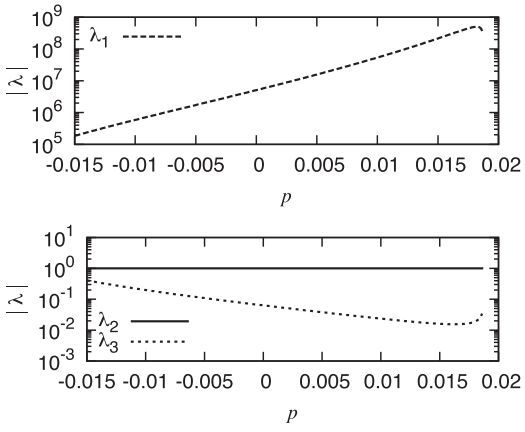
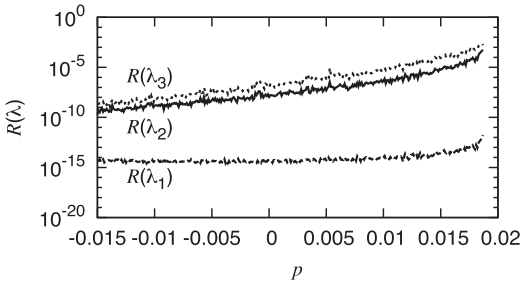


Fig. 6 The Floquet multipliers λ_j ($j = 1, 2, 3$) computed with "Method 2" for the FitzHugh-Nagumo equations (35). p: the parameter of the equations.

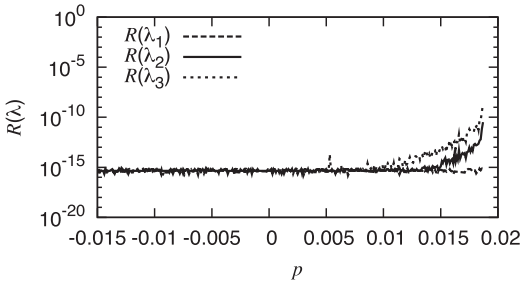
as an index of error of numerical results. Figures 8(a) and (b) compare $E(\lambda_u)$ using "Method 1" and "Method 2." We can see that $E(\lambda_u)$ can be reduced using "Method 2" with the small time increment Δt of the Runge-Kutta method, but not using "Method 1."

4.3 A Mathematical Model of Plasma Dynamics

Figures 9(a) and (b) show the computed periodic orbits in the state space and the corresponding time series of a mathematical model of plasma dynamics [13] given by

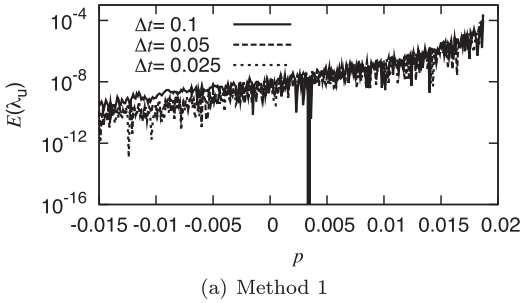


(a) Method 1

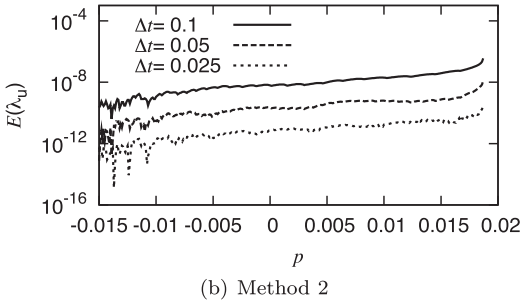


(b) Method 2

Fig. 7 Comparison of accuracy of the numerical results of the Floquet multipliers λ_j ($j = 1, 2, 3$) for the FitzHugh-Nagumo equations (35). (a) and (b) compare $R(\lambda_j)$ defined by (34) of "Method 1" and "Method 2." p: the parameter of the equations.



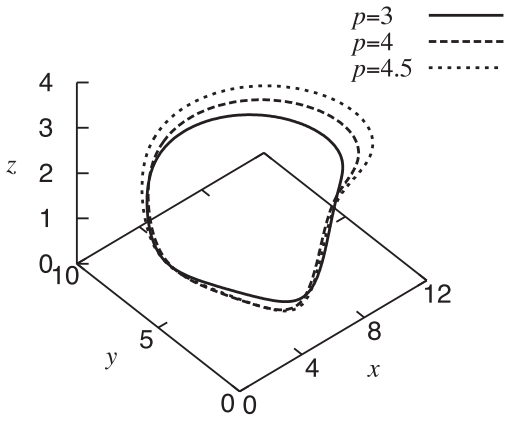
(a) Method 1



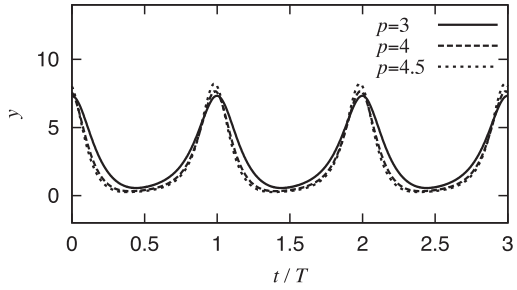
(b) Method 2

Fig. 8 Comparison of errors of the unit Floquet multiplier $\lambda_u (= \lambda_2)$ for the FitzHugh-Nagumo equations (35). p: the parameter of the equations. $E(\lambda_u)$: the error defined by (36). Δt : the time increment of the 4th order Runge-Kutta method.

$$\frac{dx}{dt} = x + y^2 \cos z\,,$$
$$\frac{dy}{dt} = -y(p + x\cos z)\,, \tag{37}$$



(a) Periodic orbits in the state space



(b) Time series of y

Fig. 9 Computed results of the periodic orbits of the model of plasma dynamics (37) by "Method 2." T: the period. p: the parameter of the model.

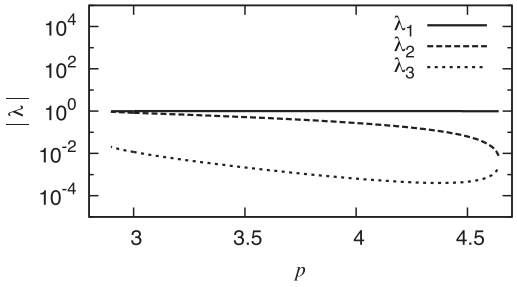
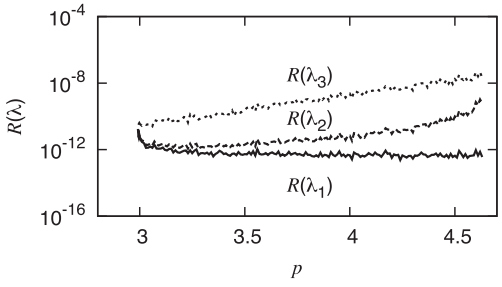


Fig. 10 The Floquet multipliers λ_j ($j = 1, 2, 3$) computed with "Method 2" for the model of plasma dynamics (37). p: the parameter of the model.

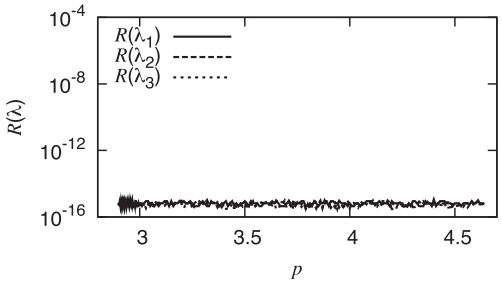
$$\frac{dz}{dt} = -q + \frac{2x^2 - y^2}{x}\sin z\,,$$

for $p = 3$, 4, and 4.5, respectively. In these computations, q was fixed to $q = 2$, $M+1 = 100$, the time increment Δt of the 4th order Runge-Kutta method was set to $\Delta t = 0.05$, and the maximum iteration number $\nu_{\max}$ of Newton's method was set to $\nu_{\max} = 15$, respectively.

Figure 10 shows the Floquet multipliers computed by "Method 2." Since this is the autonomous system, the eigenvalue λ_u ($= \lambda_1$ in this example) corresponding to unity is included. Figure 11 compares $R(\lambda_j)$ defined by (34) which can be used as an index of accuracy of numerical results. We can see that "Method 2" gives much better results than "Method 1." Figures 12(a) and (b) show the error $E(\lambda_u)$ of

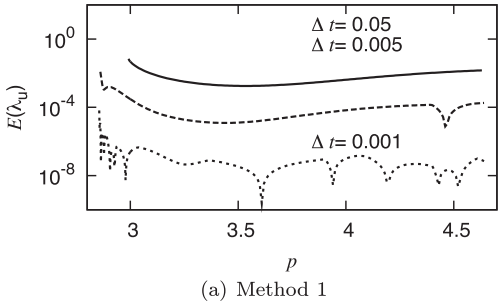


(a) Method 1

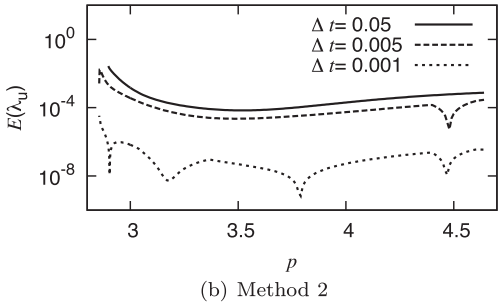


(b) Method 2

Fig. 11 Comparison of accuracy of the numerical results of the Floquet multipliers λ_j ($j = 1, 2, 3$) for the model of plasma dynamics (37). (a) and (b) compare $R(\lambda_j)$ defined by (34) of "Method 1" and "Method 2." p: the parameter of the model.



(a) Method 1



(b) Method 2

Fig. 12 Comparison of errors of the unit Floquet multiplier $\lambda_u (= \lambda_1)$ for the model of plasma dynamics (37). p: the parameter of the model. $E(\lambda_u)$: the error defined by (36). Δt: the time increment of the 4th order Runge-Kutta method.

λ_u defined by (36).

In this example, we cannot find critical differences of $E(\lambda_u)$ between "Method 1" and "Method 2." This may result from gradual time variations of the periodic orbits shown in Fig. 9(b). Then the ratios of the Floquet multipliers are not so large as those of the FitzHugh-Nagumo equations for which "Method 2" worked very effectively.

These results suggest that the proposed method "Method 2" yields more accurate Floquet multipliers than "Method 1," in particular, when the ratios of the Floquet multipliers are large. The critical point of "Method 2" is to iteratively compute eigenvectors of $\boldsymbol{X}_k(T)$ with enough accuracy. In this work, $R(\lambda_j)$ defined by (34) and $E(\lambda_u)$ defined by (36) are used as indices of error of numerical results. These indices give necessary conditions, but not sufficient conditions, for accuracy of computed Floquet multipliers. We may further investigate accuracy of numerical solutions using the proposed method, for example, following the idea of numerical verification methods [3], [14], [16]. This remains as future works.

5. Conclusions

This paper has considered a numerical method for stability analysis of periodic orbits of ordinary differential equations. Stability of periodic orbits is determined by the Floquet multipliers, namely, eigenvalues of matrix solutions of the corresponding variational equations. We have proposed a new iterative method to solve the variational equations and compute the Floquet multipliers. The keys are to utilize a property of solutions of the variational equations and to use the eigenvectors corresponding to the Floquet multipliers as initial condition for the variational equations. Numerical examples show that the proposed method works very well, in particular, when the ratios of the Floquet multipliers are large.

Acknowledgments

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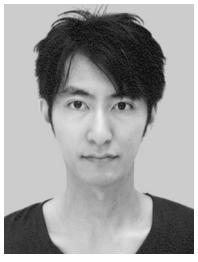

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